Numeral Analysis

Cheating Paper

Author: cht

Date: 1

2024-2025 Autumn Semester

Table of Contents

1)浮点数表示和方程求解

$\boxed{3}$ let x be the ~~interval~~ loss

a. $\frac{\cancel{150}\,x}{150} \le 10^{-3}$

$x \le 0.15$, $(149.85, 150.15)$

b. $\frac{x}{900} \le 10^{-3}$

$x \le 0.9$ $(899.1, 900.9)$

c. $\frac{x}{1500} \le 10^{-3}$

$x \le 1.5$ $(1498.5, 1501.5)$

d $\frac{x}{90} \le 10^{-3}$

$x \le 0.09$. $(89.91, 90.09)$

$\boxed{11}$ a. $\cancel{\lim\limits_{x\to 0} f(x) = \frac{x(1-\frac{1}{2}x^2+o(x^3)) - (x-\frac{1}{6}x^3+o(x^3))}{x-(x-\frac{1}{6}x^3+o(x^3))} = \lim\limits_{x\to 0} \frac{-\frac{1}{3}x^3+o(x^3)}{\frac{1}{6}x^3+o(x^3)}}$

$$= \frac{x(1-\frac{1}{2}x^2+\frac{1}{24}x^4+o(x^5)) - (x-\frac{1}{6}x^3+\frac{1}{120}x^5+o(x^5))}{x-(x-\frac{1}{6}x^3+o(x^3))}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{3}x^3+\frac{1}{30}x^5+o(x^5)}{\frac{1}{6}x^3+o(x^4)}$$

$$= -2$$

b $\cos x < 1-\frac{x^2}{2}+\frac{x^4}{4!} \Rightarrow < 0.995005$

$\cos x > 1-\frac{x^2}{2} = 0.995$

after rounding $\cos x \approx 0.9950$

$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} \approx 0.09983$

$x\cos x = 0.0995$, $x\cos x - \sin x = -0.0003300$

$x - \sin x = 0.00017$.

$\frac{x\cos x - \sin x}{x - \sin x} = -1.941$.

c. 原式 $= \dfrac{x(1-\frac{1}{2}x^2) - (x-\frac{1}{6}x^3)}{x-(x-\frac{1}{6}x^3)} = -2$

d. part(b): $\dfrac{-1.99899998-(-1.941)}{-1.99899998} = 0.029$

part(c): $\dfrac{-1.99899998-(-2)}{-1.99899998} = -0.0005$

17(b) $x_0 y_1 = 1.31 \times 4.76 = 6.2356 \approx 6.24$

$x_1 y_0 = 1.93 \times 3.24 = 6.2532 \approx 6.25$

办 $\dfrac{x_0 y_1 - x_1 y_0}{y_1 - y_0} = -0.00658 = P_1$ ①

$x_1 - x_0 = 0.620$

$(x_1 - x_0) y_0 = 2.01$

$\dfrac{(x_1-x_0)y_0}{y_1-y_0} = 1.32$

$x_0 - 1.32 = -0.0100 = P_2$ ②

$P = \dfrac{x_0 y_1 - x_1 y_0}{y_1 - y_0} = -0.0116$

$\left|\dfrac{P_1 - P}{P}\right| = 0.763 \qquad \left|\dfrac{P_2 - P}{P}\right| = 0.138$

The second method is better because its multiplication operations is more.

P38 #7

a. $\lim\limits_{h\to 0} \dfrac{\sin h}{h} = \lim\limits_{h\to 0} \dfrac{h - \frac{1}{6}h^3 + o(h^4)}{h} = 1 - \frac{1}{6}h^2 + o(h^3)$, $O(h^2)$.

b. $\lim\limits_{h\to 0} \dfrac{1-\cos h}{h} = \dfrac{1-(1-\frac{1}{2}h^2+o(h^3))}{h} = \frac{1}{2}h + o(h^2)$, $O(h)$

c. $\lim\limits_{h\to 0} \dfrac{\sin h - h\cos h}{h} = \dfrac{h - \frac{1}{6}h^3 + o(h^4) - h(1-\frac{1}{2}h^2+o(h^3))}{h}$

$\frac{1}{2}h^2 + o(h^2) = \dfrac{\frac{1}{3}h^3 + o(h^4)}{h} = \frac{1}{3}h^2 + o(h^3)$ $\quad O(h^2)$

d. $\lim\limits_{h\to 0} = \dfrac{1-[1+h+o(h)]}{h} = \cancel{1}$ $\cdot \alpha()$

$-1 - \frac{1}{2}h + o(h)$ $\quad O(h)$

2） 一元方程求解

Attention

- 这是 bisection 的特性

P54 13 为了使 $|p_n - p| \le 10^{-4}$

需 $\frac{1}{2^n}(b-a) \le 10^{-4}$

$2^n \ge 10^4$

$n \ge 14$. , $p_{14} = 1.32477$

15 此数列为调和级数，为发散

由于 $p_{2n} - p_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{1}{2}$

不满足柯西收敛准则

The order in descending speed of convergence is (b), (d), (a). The sequence in (c) does not converge.

Info

- 求不动点交点处 的导数，导数绝对值越小越收敛

Example: Find the unique root of the equation $x^3 + 4x^2 - 10 = 0$ in [1, 2]. Using the following equivalent fixed-point forms with $p_0 = 1.5$, which one is the best? (The root is approximately 1.365230013.)

a) $x = g_1(x) = x - x^3 - 4x^2 + 10$;

b) $x = g_2(x) = \sqrt{10/x - 4x}$;

c) $x = g_3(x) = \sqrt{10 - x^3}/2$;

d) $x = g_4(x) = \sqrt{10/(4+x)}$;

e) $x = g_5(x) = x - \dfrac{x^3 + 4x^2 - 10}{3x^2 + 8x}$;

OK in [1, 1.5]. $k \approx 0.66$

$k \approx 0.15$

But *why*?导数小于1

Attention

这里注意不断把函数值变成 k 乘以 x 的差 不断迭代

$\boxed{19}$. (a). $|x_n - x| = |g(x_{n-1}) - g(x)| = |g'(\xi_n)| \cdot |x_{n-1} - x| \le k|x_{n-1} - x|$

$\therefore |x_n - x| \le k|x_{n-1} - x| \le \cdots \le k^n |x_0 - x|$ $\cancel{\ge 0}$.

$\lim_{n\to+\infty} k^n = 0$, $\therefore |x_n - x| \to 0$

又. $x \doteq \sqrt{2}$, $x = \frac{1}{2}x + \frac{1}{x}$ $\therefore$收敛于$\sqrt{2}$

(b) $\cancel{x_1 = x_0}$ $\because 0 < (x_0 - \sqrt{2})^2$

$\therefore x_0^2 + 2 > 2\sqrt{2}x_0$

$\therefore x_1 = \frac{x_0^2 + 2}{2x_0} > \sqrt{2}$.

(c). i) ~~最初~~ $x_0 = \sqrt{2}$时，由于$x_1 = \frac{1}{2}x_0 + \frac{1}{x_0} = \sqrt{2}$，将恒 收敛于$\sqrt{2}$

ii) $x_0 > \sqrt{2}$ 时，根据a)，收敛于 $\sqrt{2}$

iii) $x_0 < \sqrt{2}$时 $x_1 > \sqrt{2}$，再根据 ii)，收敛于$\sqrt{2}$

3) 一元方程迭代法收敛性分析 & 矩阵求解初步

- 利用泰勒展开进行迭代计算

(1) 误差更新公式

用泰勒展开对 $g(p_n)$ 进行近似：

$$g(p_n) = g(p) + g'(p)(p_n - p) + \frac{g''(\xi_n)}{2}(p_n - p)^2,$$

其中 ξ_n 是 p_n 和 p 之间的某点。

由于 $g(p) = p$，迭代公式可写为：

$$p_{n+1} - p = g'(p)(p_n - p) + \frac{g''(\xi_n)}{2}(p_n - p)^2。$$

11. If $\frac{|p_{n+1}-p|}{|p_n-p|^3} = 0.75$ and $|p_0 - p| = 0.5$, then

$$|p_n - p| = (0.75)^{(3^n-1)/2}|p_0 - p|^{3^n}.$$

To have $|p_n - p| \le 10^{-8}$ requires that $n \ge 3$.

INPUT number of unknown and equations n; augmented matrix $A=(a_{ij})$, where $1\le i\le n$ and $1\le j\le n+1$

Output solution $x_1, x_2, \dots, x_n$ or message that the linear system has no unique solution

Step 1. For $i=1,\dots,n-1$ do Steps 2-4

Step 2 Let p be the smallest integer with $i\le p\le n$ and $a_{pi}\ne 0$

if not found. Output ('no unique solution exists'); STOP

Step 3 If $p\ne i$ then perform $(E_p)\leftrightarrow(E_i)$

Step 4 For $j=1,2,\dots,i-1,i+1,i+2,\dots,n$ do Steps 5 and 6

Step 5 Set $m_{ji}=a_{ji}/a_{ii}$

Step 6 Perform $(E_j-m_{ji}E_i)\to(E_j)$;

Step 7 If $a_{nn}=0$ then Output ('no unique solution exists'); STOP

Step 8 For $i=n-1,\dots,1$, set $x_i=[a_i/a_{ii}]$

Step 9, Output $(x_1,\dots x_n)$; STOP.

P38 #11 After an operation at line k ($1\le k\le n$), we have $(n-1)$ times multiplications / divisions, another $(n-k)(n-1)$ multiplications/divisions, $(n-k+1)(n-1)$ times additions/subtractions and n times divisions.

① $\sum_{k=1}^{n}(2n-1)+(n-k)(n-1)=\frac{n^3+2n^2-n}{2}$

② $\sum_{k=1}^{n}(n-k+1)(n-1)=\frac{n^3}{2}-\frac{n}{2}$.

P397 #7. (1) ① $\sum_{k=1}^{n}(n-k)(n-k)+n-k=\frac{1}{3}n^3-\frac{1}{3}n$

② $\sum_{k=1}^{n}(n-k)^2=\frac{1}{3}n^3-\frac{1}{2}n^2+\frac{1}{6}n$

(2) ① $\sum_{k=1}^{n}(k-1)=\frac{1}{2}n^2-\frac{1}{2}n$ ② $\sum_{k=1}^{n}(k-1)=\frac{1}{2}n^2-\frac{1}{2}n$.

(3) To solve $U\vec{x}=\vec{y}$, what we need to do is to divide E_i by U_{ii}, and then the same operations as (2), that is, $\frac{1}{2}n^2+\frac{n}{2}$ for multiplications/Divisions $\frac{1}{2}n^2-\frac{1}{2}n$ for additions / subtractions. Totally we need $\frac{1}{3}n^3+n^2-\frac{1}{3}n$ and $\frac{1}{3}n^3+\frac{1}{2}n^2-\frac{5}{6}n$ (4) all the operations for $Ly=b$ and $Ux=y$ are m times than (3), so in total $\frac{1}{3}n^3+mn^2-\frac{1}{3}n$ / $\frac{1}{3}n^3+(m-\frac{1}{2})n^2-(m-\frac{1}{6})n$

Attention
高斯约当消元 相比高斯消元的 上三角矩阵，是一个 单位矩阵

4) Choleski 分解 & 迭代线性系统解法初步

- 在上面那张图片中

17. Let

$$A = \begin{bmatrix} \alpha & 1 & 0 \\ \beta & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Find all values of α and β for which

a. A is singular.

b. A is strictly diagonally dominant.

c. A is symmetric.

d. A is positive definite.

P412 #17 (a). $|A| = 4\alpha - \alpha - 2\beta = 0$, $3\alpha = 2\beta$

(b) $\begin{cases} |\alpha| > 1 \\ |\beta| + 1 < 2 \end{cases}$ $\therefore |\alpha| > 1, |\beta| < 1.$

(c). $\beta = 1$

(d). $\begin{cases} \alpha > 0 \\ 2\alpha - 1 > 0 \\ 4\alpha - \alpha - 2 > 0 \end{cases}$ $\therefore \alpha > \frac{2}{3}$, $\beta = 1$

5) 矩阵迭代解法与插值

- 范数

矩阵范数

$\mathbf{R}^{n\times n}$上的矩阵范数是一个函数$\|\cdot\|:\mathbf{R}^{n\times n}\to\mathbf{R}$，满足下列条件：

1. $\|\mathbf{A}\|\geq 0$，且$\|\mathbf{A}\|=0$当且仅当$\mathbf{A}$是零矩阵；$(\mathbf{A}\in\mathbf{R}^{n\times n})$
2. $\|\alpha\mathbf{A}\|=|\alpha|\|\mathbf{A}\|$，其中$\alpha\in\mathbf{R},\mathbf{A}\in\mathbf{R}^{n\times n}$；
3. $\|\mathbf{A}+\mathbf{B}\|\leq\|\mathbf{A}\|+\|\mathbf{B}\|$。$(\mathbf{A},\mathbf{B}\in\mathbf{R}^{n\times n})$
4. $\|\mathbf{AB}\|\leq\|\mathbf{A}\|\|\mathbf{B}\|$。$(\mathbf{A},\mathbf{B}\in\mathbf{R}^{n\times n})$

矩阵$\mathbf{A}$和$\mathbf{B}$之间的距离定义为$\|\mathbf{A}-\mathbf{B}\|$。

$\mathbf{R}^n$上的向量范数是一个函数$\|\cdot\| : \mathbf{R}^n \to \mathbf{R}$，满足下列条件：

1. $\|\mathbf{x}\| \geq 0$，且$\|\mathbf{x}\| = 0$当且仅当$\mathbf{x} = \mathbf{0}$；($\mathbf{x} \in \mathbf{R}^n$)
2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$，其中$\alpha \in \mathbf{R}, \mathbf{x} \in \mathbf{R}^n$；
3. $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$。($\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$)

- 根据谱半径 看是否收敛

(a) 矩阵

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

- **特性：对称矩阵，正定矩阵。**
- **特征值：** 解 $\det(A - \lambda I) = 0$，得：

$$\lambda_1 = 1, \lambda_2 = 3。$$

- **谱半径：** $\rho(A) = \max\{|\lambda_1|, |\lambda_2|\} = 3$。

P436 #3. c. is convergent.

P453 #13 $T_w = (D-wL)^{-1}(1-w)D + (D-wL)^{-1}wU$

$\therefore |T_w| = |D^{-1}||(1-w)D| + |D^{-1}|w|U|$

$= |1-w|$

$= \lambda_1\lambda_2\cdots\lambda_n \leq [\rho(A)]^n = 1$

$\therefore w \in (0,2)$

6) 条件数与特征向量

一个 $n \times n$ 的希尔伯特矩阵是一个特殊矩阵，其元素定义为：

$$H_{ij} = \frac{1}{i+j-1}, \quad i, j = 1, 2, \ldots, n。$$

例如，3x3 的希尔伯特矩阵 H 为：

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0.3333 \\ 0.5 & 0.3333 & 0.25 \\ 0.3333 & 0.25 & 0.2 \end{bmatrix}。$$

#1. a. $A^{-1}=\begin{bmatrix}-18 & -24\\ -24 & 36\end{bmatrix}$, $\|A\|_\infty\|A^{-1}\|_\infty = 50$

b. $A^{-1}=\dfrac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}=\begin{bmatrix}6.744 & -3.721\\ -15.814 & 9.07\end{bmatrix}$

$\|A^{-1}\|_\infty=24.884$, $k(A)=241.37$

c. $A^{-1}=\begin{bmatrix}-100000 & 100000\\ 50000.5 & -50000\end{bmatrix}$

$\therefore \|A^{-1}\|_\infty=200000$, $k_\infty(A)=600002$

d. $A^{-1}=0.3659\times\begin{bmatrix}321.8 & -58.09\\ -5.55 & 1.003\end{bmatrix}$

$\therefore \|A^{-1}\|_\infty=\cancel{139}\ 139\ 1038.23$, $k_\infty(A)=339864.19$

e. $A^{-1}=\begin{bmatrix}1 & 1 & -2\\ 0 & 1 & -1\\ 0 & 0 & -1\end{bmatrix}$, $\|A^{-1}\|_\infty=4$

$\therefore k_\infty(A)=12$

f. $k(A)=198.17$

#9. $H=\begin{bmatrix}1 & 0.5 & 0.3333\\ 0.5 & 0.3333 & 0.25\\ 0.3333 & 0.25 & 0.2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0.5 & 1 & 0\\ 0.3333 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0.5 & 0.3333\\ 0 & 0.0833 & 0.0833\\ 0 & 0 & 0.0056\end{bmatrix}$

$H\cdot H^{-1}=I$, $\therefore LUH^{-1}=I$. 设 $Y=UH^{-1}$, $Y=\begin{bmatrix}1 & 0 & 0\\ -0.5 & 1 & 0\\ 0.1667 & -1 & 1\end{bmatrix}$

$\therefore \begin{bmatrix}1 & 0.5 & 0.3333\\ 0 & 0.0833 & 0.0833\\ 0 & 0 & 0.0056\end{bmatrix}H^{-1}=Y$, $\therefore H^{-1}=\begin{bmatrix}8.968 & -35.77 & 29.77\\ -35.77 & 190.6 & -178.6\\ 29.77 & -178.6 & 178.6\end{bmatrix}$

同理, $\hat{H}=\begin{bmatrix}0.9799 & 0.4870 & 0.3338\\ 0.4866 & 0.3246 & 0.2434\\ 0.3232 & 0.2433 & 0.1949\end{bmatrix}$

$\|H-\hat{H}\|_\infty=0.0426$

7) 拉格朗日插值+牛顿多项式插值法+Hermit 插值

- 第十七题就是要求误差在限定范围的最大步长,（线性插值）公式在 ppt 中有

P$_{n9}$ #5 $\overset{-2}{P_0}=\frac{1}{9}$, $\overset{-1}{P_1}=\frac{1}{3}$, $\overset{0}{P_2}=1$, $\overset{1}{P_3}=3$, $\overset{2}{P_4}=9$.

$P_{01}=\frac{5}{9}+\frac{2}{9}\cdot\frac{1}{2}=\frac{2}{3}$, $P_{12}=\frac{4}{3}$, $P_{23}=2$, $P_{34}=0$.

$P_{02}=\frac{3}{2}$, $P_{13}=\frac{11}{6}$, $P_{24}=-\frac{3}{2}$.

$P_{03}=\frac{16}{9}$, $P_{14}=\frac{5}{3}$

$P_{04}=\frac{41}{24}$ $\therefore \sqrt{3}\approx\frac{41}{24}$.

P120 #17 $\left|\frac{f''(\xi)}{2!}(x-(k+1)d)(x-kd)\right| \le \frac{h}{8\ln 10\,\varepsilon^2} = \frac{h}{8\ln 10}$

$h \le 0.00429$，选择 $h=0.004$ 确保整除 9

- 牛顿前向差分

基本公式

$$f(x) \approx f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 f(x_0) + \dots$$

其中:

- $x_0, x_1, x_2, \dots$ 是等距数据点，步长为 $h = x_{i+1} - x_i$。
- $p = \frac{x-x_0}{h}$，表示插值点与起始点的相对位置。
- $\Delta f(x_i)$ 是**前向差分表**中的差分值:
  - 一阶前向差分: $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$。
  - 二阶前向差分: $\Delta^2 f(x_i) = \Delta f(x_{i+1}) - \Delta f(x_i)$。
  - 三阶前向差分: $\Delta^3 f(x_i) = \Delta^2 f(x_{i+1}) - \Delta^2 f(x_i)$，以此类推。

- 后向差分

基本公式

$$f(x) \approx f(x_n) + p\nabla f(x_n) + \frac{p(p+1)}{2!}\nabla^2 f(x_n) + \frac{p(p+1)(p+2)}{3!}\nabla^3 f(x_n) + \dots$$

其中:

- $x_0, x_1, x_2, \dots, x_n$ 是等距数据点，步长为 $h = x_{i+1} - x_i$。
- $p = \frac{x-x_n}{h}$，表示插值点与末尾点的相对位置。
- $\nabla f(x_i)$ 是**后向差分表**中的差分值:
  - 一阶后向差分：$\nabla f(x_i) = f(x_i) - f(x_{i-1})$。
  - 二阶后向差分：$\nabla^2 f(x_i) = \nabla f(x_i) - \nabla f(x_{i-1})$。
  - 三阶后向差分：$\nabla^3 f(x_i) = \nabla^2 f(x_i) - \nabla^2 f(x_{i-1})$，以此类推。

- 差商 （PPT 上应该也有）

前向差商公式回顾

1. **零阶差商:**

$$f[x_0] = f(x_0),\ f[x_1] = f(x_1),\ f[x_2] = f(x_2)$$

零阶差商是直接使用函数值。

2. **一阶差商:**

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

3. **二阶差商:**

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

这些公式可以递归地从已知值推导出更高阶的差商。

8) Hermit+Cubic Spline 插值+逼近初步

- hermit 插值介绍，其中 对于 h g 函数的 也是一种估计
- 金老师的笔记里有详解，每个条件都要在多项式加上，只是前面的“项数”，即我们要估计的 函数，有时候 1 有时候 0；对于 n 个条件，要构造 n 次多项式，所以根据条件不同，要稍加变化

基于拉格朗日插值基函数✦的Hermite插值多项式方法的基本思想是构造基函数为 $h_i(x), g_i(x)$，$i=0,1 H_3(x)=y_0 h_0(x)+y_1 h_1(x)+m_0 g_0(x)+m_1 g_1(x)$ 其中 $h_i(x)$和$g_i(x)$ 在插值节点处满足相应的函数值和导数条件。具体条件见下表：

$h_0(x_0)=1$	$h_0(x_1)=0$	$h_0'(x_0)=0$	$h_0'(x_1)=0$
$h_1(x_0)=0$	$h_1(x_1)=1$	$h_1'(x_0)=0$	$h_1'(x_1)=0$
$g_0(x_0)=0$	$g_0(x_1)=0$	$g_0'(x_0)=1$	$g_0'(x_1)=0$
$g_1(x_0)=0$	$g_1(x_1)=0$	$g_1'(x_0)=0$	$g_1'(x_1)=1$

因为 $h_0(x_0)=1 \quad h_0(x_1)=0 \quad h_0'(x_0)=0 \quad h_0'(x_1)=0$ 由函数导数的零点和函数的关系，我们可以构造 $h_0(x)=(x-x_1)^2(ax+b)$ 由函数值条件 $h_0(x_0)=1, h_0'(x_0)=0$ ，将其带入 $h_0(x)$中，求解线性方程组，我们可以得到 $a=-\frac{2}{(x_0-x_1)^3}, \quad b=\frac{1}{(x_0-x_1)^2}+\frac{2x_0}{(x_0-x_1)^3}$ ，于是我们便可以得到

$$h_0(x)=\frac{(x-x_1)^2}{(x_0-x_1)^2}\left(1+\frac{2x_0}{x_0-x_1}-\frac{2x}{x_0-x_1}\right)=\left(1+2\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_1}{x_0-x_1}\right)^2=(1+2l_1(x))\cdot l_0^2(x)$$

同理

$$
\begin{aligned}
h_0(x)&=(1+2l_1(x))\cdot l_0^2(x)=\left(1+2\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_1}{x_0-x_1}\right)^2\\
h_1(x)&=(1+2l_0(x))\cdot l_1^2(x)=\left(1+2\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_0}{x_1-x_0}\right)^2\\
g_0(x)&=(x-x_0)\cdot l_0^2(x)=(x-x_0)\left(\frac{x-x_1}{x_0-x_1}\right)^2\\
g_1(x)&=(x-x_1)\cdot l_1^2(x)=(x-x_1)\left(\frac{x-x_0}{x_1-x_0}\right)^2
\end{aligned}
$$

于是将 $h_i(x), g_i(x)$，$i=0,1$ 带入插值多项式中可得

$$H_3(x)=h_0(x)+y_1h_1(x)+m_0g_0(x)+m_1g_1(x)$$

$$
\begin{aligned}
=&y_0\left(1+2l_1(x)\right)\cdot l_0^2(x)+y_1\left(1+2l_0(x)\right)\cdot l_1^2(x)+m_0\left(x-x_0\right)\cdot l_0^2(x)\\
&+m_1\left(x-x_1\right)\cdot l_1^2(x)\\
=&y_0\left(1+2\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_1}{x_0-x_1}\right)^2+y_1\left(1+2\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_0}{x_1-x_0}\right)^2\\
&+m_0\left(x-x_0\right)\left(\frac{x-x_1}{x_0-x_1}\right)^2+m_1\left(x-x_1\right)\left(\frac{x-x_0}{x_1-x_0}\right)^2
\end{aligned}
$$

于是便得到了Hermite插值多项式

The Hermite polynomial generated from these data is

$$
\begin{aligned}
H_9(x)=&\,75x+0.222222x^2(x-3)-0.0311111x^2(x-3)^2-0.00644444x^2(x-3)^2(x-5)\\
&+0.00226389x^2(x-3)^2(x-5)^2-0.000913194x^2(x-3)^2(x-5)^2(x-8)\\
&+0.000130527x^2(x-3)^2(x-5)^2(x-8)^2-0.0000202236x^2(x-3)^2(x-5)^2(x-8)^2(x-13).
\end{aligned}
$$

a. The Hermite polynomial predicts a position of $H_9(10)=743$ ft and a speed of $H_9'(10)=48$ ft/s. Although the position approximation is reasonable, the low speed prediction is suspect.

b. To find the first time the speed exceeds 55 mi/h $=80.\overline{6}$ ft/s, we solve for the smallest value of t in the equation $80.\overline{6}=H_9'(x)$. This gives $x\approx 5.6488092$.

c. The estimated maximum speed is $H_9'(12.37187)=119.423$ ft/s ≈ 81.425 mi/h.

P153 #9. $S_0(1)=1$, $\therefore B=D=\frac{1}{4}$, $b=-\frac{1}{2}$, $a=\frac{1}{4}$

$$\begin{cases} S_0(2)=S_1(2)=1 \\ S_1(3)=0. \\ S_0'(2)=S_1'(2). \\ S_0''(2)=S_1''(2) \end{cases}$$

#17
$$F(x)=\begin{cases} 20(e^{0.1}-1)x+1 & x\in[0,0.05) \\ 20(e^{0.2}-e^{0.1})x+2e^{0.1}-2e^{0.2} & x\in[0.05,.1) \end{cases}$$

$\therefore \int_0^{0.1} F(x)\,dx = 0.110795$, $\int_0^{0.1} e^{2x}\,dx = 0.1107$

- jiepeng lab 里的一个例子

特殊例子 ¶

假设 $x_0 \neq x_1 \neq x_2$，给定 $f(x_0), f(x_1), f(x_2), f'(x_1)$，找到多项式使得$P(x_i) = f(x_i)$，$P'(x_1) = f'(x_1)$。

首先，其次数为3次，我们猜想其形式为

$$P(x) = \sum_{i=0}^{2} f(x_i)h_i(x) + f'(x_1)\hat{h}_1(x)$$

其中$h_i(x_j) = \delta_i(x_j), h_i'(x_1) = 0, \hat{h}_1(x_i) = 0, \hat{h}_1'(x_1) = 1$。

根据这个猜想，我们试图构造出 $h_i(x)$ 和 $\hat{h}_1(x)$ 。

首先，我们可以用拉格朗日同样的方法构造出三次多项式$h_i(x)$，使得$h_i(x_j) = \delta_i(x_j)$，$h_i'(x_1) = 0$，$i = 0, 1, 2$。

对于$h_0(x)$，有根x_1, x_2，且因为 $h_0'(x_1) = 0$ 所以 x_1 是 $h_0(x)$ 的二重根，所以其形式为

$$h_0(x) = C_0(x - x_1)^2(x - x_2)$$

又因为$h_0'(x_0) = 1$，所以

$$h_0(x) = \frac{(x - x_1)^2(x - x_2)}{(x_0 - x_1)^2(x_0 - x_2)}$$

类似地，我们可以得到

$$h_2(x) = \frac{(x - x_0)(x - x_1)^2}{(x_2 - x_0)(x_2 - x_1)^2}$$

对于$h_1(x)$，有根x_0,x_2，都是单根。所以其形式为

$$h_1(x)=(Ax+B)(x-x_0)(x-x_2)$$

通过计算 $h_1(x_1)=1$，$h_1'(x_1)=0$，可以得到 A 和 B 的值。此处略。

然后，我们构造$\hat{h}_1(x)$，使得$\hat{h}_1(x_i)=0$，$\hat{h}_1'(x_1)=1$。对于$\hat{h}_1(x)$，有根x_0,x_1,x_2，所以

$$\hat{h}_1(x)=C(x-x_0)(x-x_1)(x-x_2)$$

又因为$\hat{h}_1'(x_1)=1$，所以可以通过计算得到 C 的值。此处略。

9) 逼近+Chebyshev 多项式

- 最小二乘法看金老师笔记
- 算了，我自己也搞一份

离散例子

可以证明，离散时的式子也是这样的。

Discussion 22: Approximate

x	1	2	3	4
y	4	10	18	26

with $y=a_0+a_1x+a_2x^2$ and $w\equiv 1$.

Solution: $\varphi_0(x)=1$, $\varphi_1(x)=x$, $\varphi_2(x)=x^2$

$$(\varphi_0,\varphi_0)=\sum_{i=1}^{4}1\cdot 1=4 \qquad (\varphi_1,\varphi_2)=\sum_{i=1}^{4}x_i\cdot x_i^2=100$$

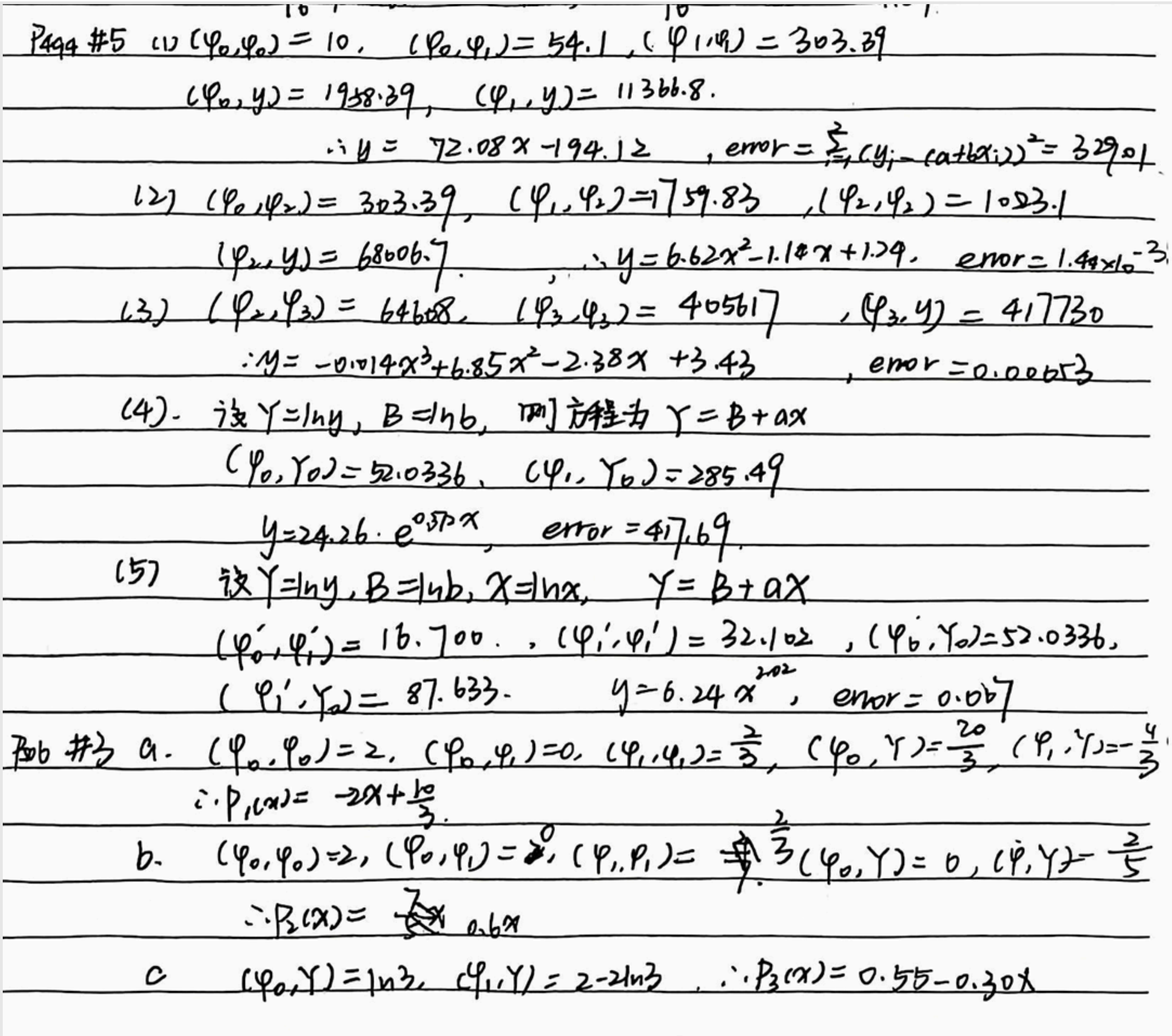
$$(\varphi_0,\varphi_1)=\sum_{i=1}^{4}1\cdot x_i=10 \qquad (\varphi_1,\varphi_1)=\sum_{i=1}^{4}x_i^2=30$$

$$(\varphi_0,\varphi_2)=\sum_{i=1}^{4}1\cdot x_i^2=30 \qquad (\varphi_2,\varphi_2)=\sum_{i=1}^{4}x_i^4=354$$

$$(\varphi_0,y)=\sum_{i=1}^{4}1\cdot y_i=58 \quad (\varphi_1,y)=182 \quad (\varphi_2,y)=622$$

$$\begin{pmatrix}4 & 10 & 30\\ 10 & 30 & 100\\ 30 & 100 & 354\end{pmatrix}\begin{pmatrix}a_0\\ a_1\\ a_2\end{pmatrix}=\begin{pmatrix}58\\ 182\\ 622\end{pmatrix} \Rightarrow \begin{array}{l} a_0=-\dfrac{3}{2},\ a_1=\dfrac{49}{10},\ a_2=\dfrac{1}{2} \\ y=P(x)=\dfrac{1}{2}x^2+\dfrac{49}{10}x-\dfrac{3}{2}\end{array}$$

$$\|B\|_\infty=484,\quad \|B^{-1}\|_\infty=\frac{63}{4} \Rightarrow K(B)=7623$$



P494 #5 (1) $(\varphi_0,\varphi_0)=10$, $(\varphi_0,\varphi_1)=54.1$, $(\varphi_1,\varphi_1)=303.39$

$(\varphi_0,y)=1958.39$, $(\varphi_1,y)=11366.8$.

$\therefore y=72.08x-194.12$, $error=\sum_{i=1}^{2}(y_i-(a+bx_i))^2=329.01$

(2) $(\varphi_0,\varphi_2)=303.39$, $(\varphi_1,\varphi_2)=1759.83$, $(\varphi_2,\varphi_2)=1023.1$

$(\varphi_2,y)=68006.7$, $\therefore y=6.62x^2-1.14x+1.24$, $error=1.44\times10^{-3}$

(3) $(\varphi_2,\varphi_3)=64608$, $(\varphi_3,\varphi_3)=405617$, $(\varphi_3,y)=417730$

$\therefore y=-0.014x^3+6.85x^2-2.38x+3.43$, $error=0.00053$

(4). 设 $Y=\ln y$, $B=\ln b$, 则方程为 $Y=B+ax$

$(\varphi_0,Y_0)=52.0336$, $(\varphi_1,Y_0)=285.49$

$y=24.26\cdot e^{0.372x}$, $error=417.69$

(5) 设 $Y=\ln y$, $B=\ln b$, $X=\ln x$, $Y=B+aX$

$(\varphi_0',\varphi_1')=16.700$, $(\varphi_1',\varphi_1')=32.102$, $(\varphi_0',Y_0)=52.0336$,

$(\varphi_1',Y_0)=87.633$. $y=6.24x^{2.02}$, $error=0.067$

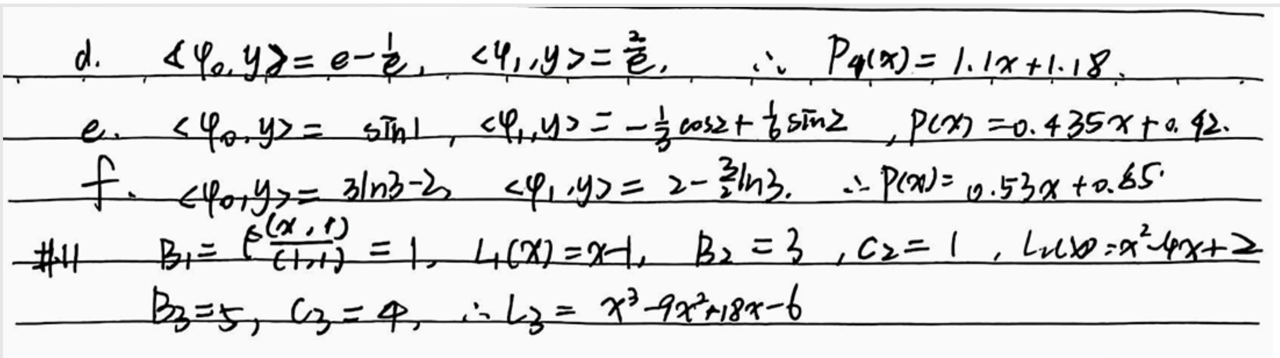
P506 #3 a. $(\varphi_0,\varphi_0)=2$, $(\varphi_0,\varphi_1)=0$, $(\varphi_1,\varphi_1)=\frac{2}{3}$, $(\varphi_0,Y)=\frac{20}{3}$, $(\varphi_1,Y)=-\frac{4}{3}$

$\therefore P_1(x)=-2x+\frac{10}{3}$.

b. $(\varphi_0,\varphi_0)=2$, $(\varphi_0,\varphi_1)=\cancel{2}\,0$, $(\varphi_1,\varphi_1)=\frac{2}{3}$, $(\varphi_0,Y)=0$, $(\varphi_1,Y)=-\frac{2}{5}$

$\therefore P_2(x)=\cancel{\frac{3}{5}x}\ 0.6x$

c $(\varphi_0,Y)=\ln 3$, $(\varphi_1,Y)=2-2\ln 3$, $\therefore P_3(x)=0.55-0.30x$



d. $\langle\varphi_0,y\rangle=e-\frac{1}{e}$, $\langle\varphi_1,y\rangle=\frac{2}{e}$, $\therefore P_4(x)=1.1x+1.18$.

e. $\langle\varphi_0,y\rangle=\sin 1$, $\langle\varphi_1,y\rangle=-\frac{1}{3}\cos 2+\frac{1}{6}\sin 2$, $P(x)=0.435x+0.42$.

f. $\langle\varphi_0,y\rangle=3\ln 3-2$, $\langle\varphi_1,y\rangle=2-\frac{3}{2}\ln 3$. $\therefore P(x)=0.53x+0.85$

#11 $B_1=\frac{(x,1)}{(1,1)}=1$, $L_1(x)=x-1$, $B_2=3$, $C_2=1$, $L_2(x)=x^2-4x+2$

$B_3=5$, $C_3=4$, $\therefore L_3=x^3-9x^2+18x-6$

- 上面这个正交化的公式如下:

我们可以构造出一系列的正交多项式。用下面定义的多项式函数集 $\{\phi_0(x), \phi_1(x), \cdots, \phi_n(x)\}$ 关于权函数 $w(x)$ 是正交的：

$$\phi_0(x)=1, \quad \phi_1(x)=x-B_1, \quad \phi_k(x)=(x-B_k)\phi_{k-1}(x)-C_k\phi_{k-2}(x), \quad k=2,3,\cdots$$

其中 B_k 和 C_k 是常数，可以通过

$$B_k=\frac{\langle x\phi_{k-1}, \phi_{k-1}\rangle}{\langle \phi_{k-1}, \phi_{k-1}\rangle}, \quad C_k=\frac{\langle x\phi_{k-1}, \phi_{k-2}\rangle}{\langle \phi_{k-2}, \phi_{k-2}\rangle}$$

来计算。

Example: Approximate

x	1	2	3	4
y	4	10	18	26

with $y=c_0+c_1x+c_2x^2$ and $w\equiv 1$.

Solution: First construct the orthogonal polynomials $\varphi_0(x)$, $\varphi_1(x)$, $\varphi_2(x)$.

Let $y=a_0\varphi_0(x)+a_1\varphi_1(x)+a_2\varphi_2(x)$

$$a_k=\frac{(\varphi_k, y)}{(\varphi_k, \varphi_k)}$$

$$\varphi_0(x)=1 \qquad a_0=\frac{(\varphi_0, y)}{(\varphi_0, \varphi_0)}=\frac{29}{2}$$

$$B_1=\frac{(x\varphi_0, \varphi_0)}{(\varphi_0, \varphi_0)}=\frac{5}{2} \qquad \varphi_1(x)=(x-B_1)=x-\frac{5}{2} \qquad a_1=\frac{(\varphi_1, y)}{(\varphi_1, \varphi_1)}=\frac{37}{5}$$

$$B_2=\frac{(x\varphi_1, \varphi_1)}{(\varphi_1, \varphi_1)}=\frac{5}{2} \qquad C_2=\frac{(x\varphi_1, \varphi_0)}{(\varphi_0, \varphi_0)}=\frac{5}{4}$$

$$\varphi_2(x)=(x-\frac{5}{2})\varphi_1(x)-\frac{5}{4}\varphi_0(x)=x^2-5x+5 \qquad a_2=\frac{(\varphi_2, y)}{(\varphi_2, \varphi_2)}=\frac{1}{2}$$

$$\Rightarrow \quad y=\frac{29}{2}\times 1+\frac{37}{5}(x-\frac{5}{2})+\frac{1}{2}(x^2-5x+5)=\frac{1}{2}x^2+\frac{49}{10}x-\frac{3}{2}$$

里面各项已经在之前的图片中计算过了。

10) chebyshev 多项式+数值微分与积分

Error

这里的 #7 有点问题，应该一个个降低，直到遇到不精确之后退回上一个版本

P57. #3. $x_k = \cos\left(\frac{2k-1}{2n}\pi\right), k=1,2,3,4, n=4$

$\therefore x_1 = 0.9239, \ x_2 \approx 0.3827, \ x_3 \approx -0.3827, \ x_4 \approx -0.9239$

$\therefore$ a. $P_3(x) = 2.5190 + 1.945(x-0.9239) + 0.7047(x-0.9239)(x-0.3827)$
$+0.1751(x-0.9239)(x-0.3827)(x+0.3827)$

b. $P_3(x) = 0.7979 + 0.7844(x-0.9239) - 0.1464(x-0.9239)(x-0.3827)$
$-0.1585(x-0.9239)(x-0.3827)(x+0.3827)$

c. $P_3(x) = 1.0730 + 0.3782(x-0.9239) - 0.0980(x-0.9239)(x-0.3827)$
$0.0491(x-0.9239)(x-0.3827)(x+0.3827)$

d. $P_3(x) = 0.7286 + 1.3066(x-0.9239) + 1.0000(x-0.9239)(x-0.3827)$

#7. $\sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120}$

$|R_n(x)| \leq \frac{1}{2^n(n+1)!} < \frac{1}{100}, \quad n \geq 3$

$T_0(x)=1, T_1(x)=x, T_2(x)=2x^2-1, T_3(x)=4x^3-3x$

$T_4(x) = 8x^4-8x^2+1, \ T_5(x)=16x^5-20x^3+5x$

$\therefore$ 原式 $= P_3 = P_4 - Q_4 = \frac{383}{384}x - \frac{5}{32}x^3.$

#9 $\int_{-1}^{1} \frac{[T_n(x)]^2}{\sqrt{1-x^2}}dx = \int_0^{\pi} \frac{[\cos(n\arccos(x))]^2}{\sin\theta}\sin\theta\, d\theta = \int_0^{\pi}[\cos(n\theta)]^2 d\theta$

$= \int_0^{\pi}\frac{1}{2}d\theta + \int_0^{\pi}\frac{\cos(2n\theta)}{2}d\theta = \frac{1}{2}\pi$

Attention

一是用切比雪夫逼近，二是用其降低阶数，在 notebook 中(切比雪夫多项式有两种等价的定义)

- 三点法和五点法

三点中心差分公式用于计算一阶导数 $f'(x)$：

$$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}.$$

其误差来源于泰勒展开的高阶项，误差阶数为 **二阶**，具体形式为：

$$\text{误差} = -\frac{h^2}{6}f^{(3)}(\xi),$$

其中 $\xi \in [x-h, x+h]$。

五点中心差分公式用于计算一阶导数 $f'(x)$:

$$f'(x) \approx \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}.$$

其误差来源于泰勒展开的高阶项，误差阶数为 **四阶**，具体形式为:

$$\text{误差} = -\frac{h^4}{30} f^{(5)}(\xi),$$

其中 $\xi \in [x-2h, x+2h]$。

P176 #7 $f'(3) = \frac{-f(5)+8f(4)-8f(2)+f(1)}{12} = 0.2106.$

error $\leq \frac{1^4}{30} \times 23 = 0.767.$

#13. $f''(0.5) = \frac{1}{(\frac{1}{4})^2}\left[\frac{\sqrt{2}}{2} - 0 + (-\frac{\sqrt{2}}{2})\right] = 0.$

$f^{(4)}(x) = \pi^4 \cos \pi x$, error $< \frac{1}{24 \times 16} \times 2\pi^4 = 0.51.$

because the function is symmetric at 0.5.

11) 数值积分

P195 #7. given that

$$\begin{cases} \frac{2-0}{2} \times (f(0)+f(2)) = 4 \\ \frac{1}{3}(f(0)+4f(1)+f(2)) = 2 \end{cases}$$

$$\therefore \cancel{f(1) = -\frac{1}{4}} \quad f(1) = \frac{1}{2}$$

#9 对于 $f(x)=1$, $\int_{-1}^{1} 1\,dx = f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}) = 2$

$f(x)=x$, $\int_{-1}^{1} x\,dx = f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}) = 0$

$f(x)=x^2$, $\int_{-1}^{1} x^2\,dx = f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}) = \frac{2}{3}$

$f(x)=x^3$, $\int_{-1}^{1} x^3\,dx = f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}) = 0$

$f(x)=x^4$, $\int_{-1}^{1} x^4\,dx = \frac{2}{5}$, $f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}) = \frac{2}{9}$

故精度为 3

#11

$$\begin{cases} 2 = c_0 + c_1 + c_2 & (f(x)=1) \\ 0 = -c_0 + c_2 & (f(x)=x) \\ \frac{2}{3} = c_0 + c_2 & (f(x)=x^2) \end{cases} \quad \therefore c_0 = \frac{1}{3}, c_1 = \frac{4}{3}, c_2 = \frac{1}{3}$$

#13.

$$\begin{cases} f(x)=1: & 1 = c_0 + c_1 \\ f(x)=x: & \frac{1}{2} = c_1 x_1 \\ f(x)=x^2: & \frac{1}{3} = c_1 x_1^2 \end{cases} \Rightarrow \begin{cases} c_0 = \frac{1}{4} \\ c_1 = \frac{3}{4} \\ x_1 = \frac{2}{3} \end{cases}$$

but $f(x)=x^3$, $\int_0^1 x^3\,dx = \frac{1}{4}$, $c_0 f(0) + c_1 f(x_1) = \frac{2}{9}$.

Info

复合数值积分,要求最大的步长（等价于最少的步数）

Determine the values of n and h required to approximate

$$\int_0^2 e^{2x} \sin 3x \, dx$$

to within 10^{-4}.

a. Use the Composite Trapezoidal rule.

b. Use the Composite Simpson's rule.

P204 #7 (a) $f''(x) = 12e^{2x}\cos 3x - 5e^{2x}\sin 3x \geq 705.36$

when $\frac{(\frac{2}{n})^2}{12} \times 2 \times 705.36 \leq 10^{-4}$. $n \geq 2168.5$

(b) $f^{(4)}(x) = -119e^{2x}\sin 3x - 156e^{2x}\cos 3x \geq -4475.41$

$\therefore n \geq 54$.

Example: Approximate $\int_0^1 \sqrt{x} f(x)dx$ using Gaussian quadrature with $n = 1$.

Solution: Assume $\int_0^1 \sqrt{x} f(x)dx \approx A_0 f(x_0) + A_1 f(x_1)$

***Step 1*:** Construct the orthogonal polynomial φ_2

Let $\varphi_0(x) = 1$, $\varphi_1(x) = x + a$, $\varphi_2(x) = x^2 + bx + c$

$$(\varphi_0, \varphi_1) = 0 \Rightarrow \int_0^1 \sqrt{x}(x + a)dx = 0 \Rightarrow a = -\frac{3}{5}$$

$$(\varphi_0, \varphi_2) = 0 \Rightarrow \int_0^1 \sqrt{x}(x^2 + bx + c)dx = 0$$

$$(\varphi_1, \varphi_2) = 0 \Rightarrow \int_0^1 \sqrt{x}(x - \frac{3}{5})(x + bx + c)dx = 0$$

$$\Rightarrow b = -\frac{10}{9}, \quad c = \frac{5}{21}$$

$$\varphi_2(x) = x^2 - \frac{10}{9}x + \frac{5}{21}$$

***Step 2*:** Find the two roots of φ_2 which are the Gaussisan points x_0 and x_1

$$x_{0;1} = \frac{10/9 \pm \sqrt{(10/9)^2 - 20/21}}{2}$$

***Step 3*:** Since the formula must be accurate for $f(x) = 1, x$, we can easily solve a *linear* system of equations for A_0 and A_1.

The results are the same as we have obtained:

$$x_0 \approx 0.8212,\ x_1 \approx 0.2899,\ A_0 \approx 0.3891,\ A_1 \approx 0.2776$$

Use this formula to approximate $\int_0^1 \sqrt{x} e^x dx$

$$\int_0^1 \sqrt{x} e^x dx \approx A_0 e^{x_0} + A_1 e^{x_1} = 0.3891 \times e^{0.8212} + 0.2776 \times e^{0.2899} \approx 1.2555$$

$\int_0^1 \sqrt{x}(2x - 1)dx = 2/15$ Accurate!

Attention

A_1A_2的求法, 让 fx=1 fx=x 得到两个等式

$$\int_0^1 \sqrt{x}f(x)dx \approx A_0f(x_0) + A_1f(x_1)$$

12) 微分方程数值解法：显式法与隐式法

P271 #5 (a) $\begin{cases} \phi(t,w) = f(t,w) + \frac{h}{2} f'(t_i, w_i) \\ w_{i+1} = w_i + h\phi(t_i, w_i) \end{cases}$

而 $f'(t,y) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f(t,y)$，而 $\begin{cases} \frac{\partial f}{\partial t} = -\frac{2}{t^2}y + (2te^t + t^2 e^t) \\ \frac{\partial f}{\partial y} = \frac{2}{t} \end{cases}$

初始：$t_0 = 1, w_0 = 0$，

$\therefore t = 1.5$，numerical $y(w) = 3.91098$，Exact $y = 3.96767$

$t = 2.0$，$y(w) = 18.46999$　　$y = 18.683097$

(b) $\because y(t) \approx w_i + \frac{w_{i+1} - w_i}{t_{i+1} - t_i}(t - t_i)$

$y(1.04) \approx 0.1368914$，$y(1.55) \approx 4.777233$，$y(1.97) \approx 17.174801$

P301 #10. 已知 $y''(t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f(t,y)$

$y'''(t) = \frac{\partial^2 f}{\partial t^2} + 2\frac{\partial^2 f}{\partial t \partial y} \cdot f + \frac{\partial f}{\partial y} \cdot y'' + \frac{\partial^2 f}{\partial y^2} f^2$

展开式：$f(t_{i-1}, y(t_{i-1})) = f(t_i, y(t_i)) - h\frac{\partial f}{\partial t} - h\frac{\partial f}{\partial y} \cdot f(t_i, y(t_i))$
$+ \frac{h^2}{2}\frac{\partial^2 f}{\partial t^2} + O(h^3)$.

$f(t_{i-2}, y(t_{i-2})) = f(t_i, y(t_i)) - 2h\frac{\partial f}{\partial t} - 2h\frac{\partial f}{\partial y} \cdot f(t_i, y(t_i)) + 2h^2\frac{\partial^2 f}{\partial t^2} + O(h^3)$

代入 $y(t_{i+1}) = y(t_i) + ahf(t_i, y(t_i)) + bhf(t_{i-1}, y(t_{i-1})) + chf(t_{i-2}, y(t_{i-2}))$

整理后，$y(t_{i+1}) = y(t_i) + h[af(t_i, y(t_i)) + bf(t_i, y(t_i)) + cf(t_i, y(t_i))]$
$+ h^2[-b\frac{\partial f}{\partial t} - b\frac{\partial f}{\partial y} f(t_i, y(t_i)) - 2c\frac{\partial f}{\partial t} - 2c\frac{\partial f}{\partial y} f(t_i, y(t_i))$
$+ h^3[\frac{b}{2}\frac{\partial^2 f}{\partial t^2} + 2c\frac{\partial^2 f}{\partial t^2}]$

$\therefore \begin{cases} a + b + c = 1 \\ -b - 2c = \frac{1}{2} \\ \frac{b}{2} + 2c = \frac{1}{6} \end{cases}$　$\therefore \begin{cases} a = \frac{23}{12} \\ b = -\frac{4}{3} \\ c = \frac{5}{12} \end{cases}$

$\therefore y(t_{i+1}) = y(t_i) + \frac{23}{12} hf(t_i, y(t_i)) - \frac{4}{3} hf(t_{i-1}, y(t_{i-1})) + \frac{5}{12} hf(t_{i-2}, y(t_{i-2}))$

从泰勒展开公式:

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2!}y''(t_i) + \frac{h^3}{3!}y^{(3)}(t_i) + O(h^4)$$

注意:

- $y'(t_i) = f(t_i, y(t_i))$
- 根据微分方程，可以写出:

$$y''(t_i) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}y'(t_i) = f_t + f_y f$$

$$y^{(3)}(t_i) = \frac{\partial^2 f}{\partial t^2} + 2\frac{\partial^2 f}{\partial t \partial y}f + \frac{\partial^2 f}{\partial y^2}f^2 + \frac{\partial f}{\partial y}(f_t + f_y f)$$

2. 展开 $f(t_{i-1}, y(t_{i-1}))$ 和 $f(t_{i-2}, y(t_{i-2}))$:

类似地，对 $f(t_{i-1}, y(t_{i-1}))$ 进行泰勒展开:

$$f(t_{i-1}, y(t_{i-1})) = f(t_i, y(t_i)) - hf_t - hf_y f + \frac{h^2}{2!}(f_{tt} + 2f_{ty}f + f_{yy}f^2) + O(h^3)$$

对于 $f(t_{i-2}, y(t_{i-2}))$，由于时间间隔是 $2h$，展开如下:

$$f(t_{i-2}, y(t_{i-2})) = f(t_i, y(t_i)) - 2hf_t - 2hf_y f + \frac{(2h)^2}{2!}(f_{tt} + 2f_{ty}f + f_{yy}f^2) + O(h^3)$$

13) 微分方程数值解法：Runge-Kutta 法和稳定性

Attention

注意，每一个 K_{n+1}都依赖于 K_n，除非是第一个

第⑩题的K_4回答有误

P280 #1 $t=0.5$时, $y_{pre} = y_0 + h\cdot f(t_0, y_0) = 0 + 0.5\times 0 = 0$

$\therefore y_1 = y_0 + \frac{h}{2}[f(t_0, y_0) + f(t_1, y_{predict})]$
$= \frac{1}{8}e^{\frac{3}{2}}$

$y_{pre}' = y_1 + h\cdot f(t_1, y_1) = \frac{1}{4}e^{\frac{3}{2}}$

$\therefore y_2 = y_1 + \frac{h}{2}[f(t_1, y_1) + f(t_2, y_{pre}')] = \frac{1}{16}e^{\frac{3}{2}} + \frac{1}{4}e^{3}$

	approximation	real
$t=0.5$	0.560211	0.283616.5
$t=1$	~~0.~~5.30149.	3.219099.

#10. $t=0.5$时, $k_1 = f(t_0, y_0) = f(0,0) = 0.$

$k_2 = f(t_0+\frac{h}{2}, y_0+\frac{h}{2}k_1) = f(\frac{1}{4}, 0) = \frac{1}{4}e^{\frac{3}{4}}$

$k_3 = f(t_0+\frac{h}{2}, y_0+\frac{h}{2}k_2) = f(\frac{1}{4}, \frac{1}{16}e^{\frac{3}{4}}) = \frac{1}{8}e^{\frac{3}{4}}.$

$k_4 = f(t_0+\frac{h}{2}, y_0+\frac{h}{2}k_3) = f(\frac{1}{4}, \frac{1}{8}e^{\frac{3}{4}}) = 0$

$y_{\frac{1}{2}} = y_0 + \frac{1}{12}(k_1+2k_2+2k_3+k_4) = \frac{1}{16}e^{\frac{3}{4}}.$

$t=1$时 $k_1' = f(\frac{1}{2}, \frac{1}{16}e^{\frac{3}{4}}) = \cancel{\frac{3}{8}e^{\frac{3}{4}}}\ \frac{1}{2}e^{\frac{3}{2}} - \frac{1}{8}e^{\frac{3}{4}}$

$k_2' = f(t_0+\frac{h}{2}, y_1+\frac{h}{2}k_1) = f(\frac{3}{4}, \frac{5}{32}e^{\frac{3}{4}}) = \frac{3}{4}e^{\frac{9}{4}} - \cancel{\frac{5}{16}e^{\frac{3}{4}}}$

$k_3' = f(t_0+\frac{h}{2}, y_1+\frac{h}{2}k_2) = f(\frac{3}{4}, \frac{3}{16}e^{\frac{9}{4}} - \frac{1}{64}e^{\frac{3}{4}}) = \frac{9}{128}e^{\frac{3}{4}} + \frac{3}{32}e^{\frac{9}{4}}$

$k_4' = f(t_0+\frac{h}{2}, y_1+\frac{h}{2}k_3) = f(\frac{3}{4}, \frac{9}{128}e^{\frac{3}{4}} + \frac{3}{32}e^{\frac{9}{4}}) = \frac{9}{16}e^{\frac{9}{4}} - \frac{9}{64}e^{\frac{3}{4}}$

~~$k_2' = f(\frac{3}{4}, \frac{1}{8}e^{\frac{3}{2}} + \frac{1}{32}e^{\frac{3}{4}}) =$~~

approximation: $t=0.5$, $w_1 = 0.264625$, $t=1$, $w_2 = 1.48348.$

#13. 中点法: $k_1 = f(t_n, y_n) = -y_n + t_n + 1$

$k_2 = f(t_n+\frac{h}{2}, y_n+\frac{h}{2}k_1) = -y_n - \frac{h}{2}k_1 + t_n + \frac{h}{2} + 1 = -y_n + \frac{h}{2}y_n - \frac{h}{2}t_n + t_n + 1$

$y_{n+1} = y_n + h\cdot k_2 = -hy_n + \frac{1}{2}h^2 y_n - \frac{1}{2}h^2 t_n + ht_n + h + y_n$

修正欧拉法: $k_1 = -y_n + t_n + 1$; $y_{pre} = y_n - hy_n + ht_n + h$

$k_2 = -y_n + hy_n - ht_n + t_n + 1$, $\therefore y_{n+1} = -hy_n + \frac{1}{2}h^2 y_n - \frac{1}{2}h^2 t_n + ht_n + h + y_n$

何因. $y_{n+1} = y_n + \frac{h}{2}(f(t_n, y_n) + f(t_n+h, y_{pre}))$ (与上一种方法相似)

$= -hy_n + \frac{1}{2}h^2 y_n - \frac{1}{2}h^2 t_n + ht_n + h + y_n$

原因: $f(t_n, y)$ 是线性函数

Attention
Adams Fourth-Order Predictor-Corrector Algorithm

1. **初始化**:
   - 使用单步法（如四阶 Runge-Kutta 方法）计算前四个点的值 y_0, y_1, y_2, y_3。这是因为 Adams 方法需要至少 4 个点来启动。
2. **Adams-Bashforth Predictor**:
   - 使用四阶 Adams-Bashforth 公式预测下一个点 y_{pred}。
3. **Adams-Moulton Corrector**:
   - 使用预测值 y_{pred}，应用四阶 Adams-Moulton 公式修正得到 y_{corr}。
4. **更新**:
   - 将修正值 y_{corr} 作为当前点的最终值，并进入下一步。
5. **重复**:
   - 依次使用 Adams-Bashforth 方法预测，Adams-Moulton 方法校正，直到覆盖整个区间。

Example

1. 初始化 (使用 Runge-Kutta 方法计算前 4 个点)

Adams 方法需要至少 4 个初始点，因此我们使用 **四阶 Runge-Kutta 方法 (RK4)** 来计算 y_1, y_2, y_3, y_4。

RK4 的公式:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

其中:

$$k_1 = hf(t_n, y_n), \quad k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right), \quad k_3 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right), \quad k_4 = hf(t_n + h, y_n + k_3)$$

初始条件:

$$t_0 = 0,\ y_0 = 1$$

计算第 1 步:

$$t_1 = 0.2, \quad h = 0.2$$

$$k_1 = hf(t_0, y_0) = 0.2 \cdot (-2 \cdot 0 \cdot 1) = 0$$

$$k_2 = hf(t_0 + h/2, y_0 + k_1/2) = 0.2 \cdot (-2 \cdot 0.1 \cdot 1) = -0.04$$

$$k_3 = hf(t_0 + h/2, y_0 + k_2/2) = 0.2 \cdot (-2 \cdot 0.1 \cdot 0.98) = -0.0392$$

$$k_4 = hf(t_0 + h, y_0 + k_3) = 0.2 \cdot (-2 \cdot 0.2 \cdot 0.9608) = -0.076864$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + \frac{1}{6}(0 - 0.08 - 0.0784 - 0.076864) \approx 0.96079$$

类似计算 y_2, y_3, y_4，得到：

$$t_2 = 0.4,\ y_2 \approx 0.85214$$

$$t_3 = 0.6,\ y_3 \approx 0.69767$$

$$t_4 = 0.8,\ y_4 \approx 0.51679$$

---

2. 开始使用 Adams 方法

我们已经得到了前 4 个点：$(t_0, y_0), (t_1, y_1), (t_2, y_2), (t_3, y_3), (t_4, y_4)$。

接下来，使用 **Adams-Bashforth Predictor** 预测 y_{pred}，然后用 **Adams-Moulton Corrector** 修正。

Adams-Bashforth Predictor (预测公式)

公式：

$$y_{\text{pred}} = y_n + h \cdot \frac{1}{24}\left(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right)$$

对 $t_4 = 0.8, y_4 = 0.51679$ 预测 y_{pred}：

$$f_4 = f(t_4, y_4) = -2 \cdot 0.8 \cdot 0.51679 = -0.82786$$

$$f_3 = f(t_3, y_3) = -2 \cdot 0.6 \cdot 0.69767 = -0.83697$$

$$f_2 = f(t_2, y_2) = -2 \cdot 0.4 \cdot 0.85214 = -0.68171$$

$$f_1 = f(t_1, y_1) = -2 \cdot 0.2 \cdot 0.96079 = -0.38432$$

代入预测公式：

$$y_{\text{pred}} = y_4 + h \cdot \frac{1}{24}\left(55f_4 - 59f_3 + 37f_2 - 9f_1\right)$$

$$y_{\text{pred}} = 0.51679 + 0.2 \cdot \frac{1}{24}\left(55(-0.82786) - 59(-0.83697) + 37(-0.68171) - 9(-0.38432)\right)$$

$$y_{\text{pred}} \approx 0.51679 - 0.2 \cdot 0.03426 = 0.51094$$

Adams-Moulton Corrector (校正公式)

公式:

$$y_{\text{corr}} = y_n + h \cdot \frac{1}{24}\left(9f_{\text{pred}} + 19f_n - 5f_{n-1} + f_{n-2}\right)$$

计算预测点的斜率:

$$f_{\text{pred}} = f(t_5, y_{\text{pred}}) = -2 \cdot 1.0 \cdot 0.51094 = -1.02188$$

代入校正公式:

$$y_{\text{corr}} = y_4 + h \cdot \frac{1}{24}\left(9(-1.02188) + 19(-0.82786) - 5(-0.83697) + (-0.68171)\right)$$

$$y_{\text{corr}} \approx 0.51679 - 0.2 \cdot 0.03638 = 0.50951$$

---

3. 结果

在 $t = 1.0$ 时，数值解为:

$$y(1.0) \approx 0.50951$$

如果需要更高精度，可以减小步长 h。通过重复预测-校正步骤，可以依次得到整个区间内的近似解。

7. 使用线性模型 $f(t,w)=\lambda w$.

代入公式,得. $w_{i+1} = -4w_i + 5w_{i-1} + 2h[\lambda w_i + 2h\lambda w_{i-1}]$

$$w_{i+1} = (-4+2h\lambda)w_i + (5+4h^2\lambda)w_{i-1}$$

$$\begin{bmatrix} w_{i+1} \\ w_i \end{bmatrix} = \begin{bmatrix} -4+2h\lambda & 5+4h^2\lambda \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_i \\ w_{i-1} \end{bmatrix}$$

$$\det\left(\begin{bmatrix} -4+2h\lambda-\rho & 5+4h^2\lambda \\ 1 & -\rho \end{bmatrix}\right) = 0$$

$$(-4+2h\lambda-\rho)(-\rho) - (5+4h^2\lambda) = 0$$

$$\rho^2 + (4-2h\lambda)\rho - (5+4h^2\lambda) = 0$$

无法使 $|\rho| \le 1$, 方法不稳定.